Entanglement of Collaboration

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The entanglement of collaboration (EoC) quantifies the maximum amount of entanglement, that can be generated between two parties, A and B, given collaboration with $N-2$ other parties, when the N parties share a multipartite (possibly mixed) state and where the collaboration consists of local operations and classical communication (LOCC) by *all* parties. The localizable entanglement (LE) is defined similarly except that A and B do not participate in the effort to generate bipartite entanglement. We compare between these two operational definitions and find sufficient conditions for which the EoC is equal to the LE. In particular, we find that the two are equal whenever they are measured by the concurrence or by one of its generalizations called the G-concurrence. We also find a simple expression for the LE in terms of the Jamiolkowski isomorphism and prove that it is convex.

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Introduction

Entanglement, and in particular, bipartite entanglement has been recognized as a valuable resource for important quantum information processing tasks, such as teleportation [1] and superdense coding [2]. In particular, when a quantum system shared by spatially separated parties, entanglement is a resource with which the restriction to local operations and classical communications (LOCC) can be overcome. Further restrictions on the amount and/or direction in which the classical messages are exchanged by the different parties give rise to different types of entanglement. For example, consider the distillable entanglement [3]; that is, the amount of Bell states (singlets) that can be distilled from a bipartite state, ρ, in the asymptotic limit of many copies. If the state ρ is pure then optimal distillation doesn't require any classical communication. However, if ρ is mixed then there are at least two types of distillable entanglement measures corresponding to 1-way and 2-way classical channels.

In this paper, we discuss the difference between 1-way and 2-way classical channels in the context of entanglement of assistance (EoA) [4, 5, 6] and localizable entanglement (LE) [7, 8]. EoA quantifies the entanglement that can be generated between two parties, Alice and Bob, given assistance from a third party, Charlie, when the three share a tripartite state and where the assistance consists of Charlie initially performing a measurement on his share and communicating the result to Alice and Bob through a one-way classical channel. After Alice and Bob receive the message from Charlie they end up with more entanglement then they had initially. Thus, we can view this operational definition as a method to *lock* (or more precisely, unlock) bipartite entanglement in tripartite states, where Charlie holds the classical key to unlock it. Similarly, the generalization of EoA to more then three parties, i.e. the LE, can be viewed as locking bipartite entanglement in multipartite states. In this paper, the term LE refers also to the EoA.

In [9], a slightly different operational definition has been discussed, dubbed the entanglement of collaboration (EoC), in which Alice and/or Bob are allowed to perform measurements and announce the outcome prior to the measurements performed by Charlie. It has been found that with this collaborative tripartite LOCC it is possible to increase the amount of entanglement that can be unlocked by Charlie. In what follows, we compare between these two scenarios, and find, somewhat surprisingly, that the EoC of a multipartite *mixed* state is *equal* to the LE whenever the entanglement between Alice and Bob is measured with the concurrence [10] or with the G-concurrence [11, 12]; this is despite the fact that the EoC can be strictly greater than the LE when measured with the entropy of entanglement [13]. This result is very important since the concurrence have been used widely in the study of LE, especially, in spin chains [7, 14, 15, 16, 17, 18, 19].

The localizable entanglement is not an entanglement monotone

Let us first describe, in a simple way, the example given in [9] for which the EoC is greater than the LE. In this example, we consider three parties, Alice, Bob and Charlie sharing an an $8\times 4\times 2$ tripartite pure state. The quantum state can be constructed as follows [20]: consider classical information as a message $y=0,1$ that is encoded in a basis $\{|y\rangle_C\}$. This classical information can be locked by applying one of two unitaries $\{V_x\}$, where $x=0,1$. Imagine that Charlie holds the token of the classical message, and Alice holds the key to unlocking it. Furthermore, imagine that Alice and Bob share a

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4×4 maximally entangled state, and a unitary on Bob's system is controlled by x and y. That is, the state is

$$|\Psi\rangle = \frac{1}{2}\sum_{x=0}^{1}\sum_{y=0}^{1}|x\rangle_a (I\otimes U_{xy})|\phi^+\rangle_{AB} V_x|y\rangle_C \ , \qquad (1)$$

where Alice holds systems a and A, and $|\Phi^+\rangle = (|00\rangle + |11\rangle + |22\rangle + |33\rangle)/2$ is the 4×4 maximally entangled state. The intuition behind this construction is that Charlie holds the key to getting the entanglement out of Alice and Bob, but it is locked with information that only Alice can supply. Note that if Alice measure x and send the result to Charlie, then Charlie can measure y and Alice and Bob end up with the maximally entangled state $(I \otimes U_{xy})|\phi^+\rangle_{AB}$. Thus, the EoC is two ebits and it is independent of the choice of the unitaries $\{V_x\}$ and $\{U_{xy}\}$. The LE, on the other hand, does depend on the choice of $\{V_x\}$ and $\{U_{xy}\}$, and can be less than 2 ebits. The pure state example in [9], for which the LE (i.e. EoA) is strictly less then 2 ebits, can be written in the same form as in Eq. (1) with, $V_0 = I_{2\times2}$, the identity 2×2 matrix, $V_1 = (I_{2\times2}+\sigma_y)/\sqrt{2}$, where σ_y is the second Pauli matrix, and the four unitaries U_{xy} are diagonal with $U_{x0} = I_{4\times4}$ for $x = 0, 1$, and $U_{01} = \mathrm{diag}(i, 1, -i, -1)$, $U_{11} = \mathrm{diag}(i, 1, i, 1)$. In fact, for this example, it has been shown in [9] that Charlie can not create 2 ebits between Alice and Bob even with some probability less than one.

Since the dimension of Alice-Bob system is 8×4, the example above does not rule out the possibility that the LE is a monotone for lower dimensions or when the entanglement between Alice and Bob is measured with a monotone that can not distinguish maximally entangled states from non-maximally entangled states. Indeed, among other things, we show here that for some monotones of this sort, the LE is an entanglement monotone and therefore equal to the EoC.

The examples in [9] and Eq. (1) also demonstrate the advantage of 2-Way classical channels over 1-Way channels in the process of generating entanglement between distant parties. To see that, consider a chain of n copies of the state (1) shared by $3n$ parties A_k, B_k and C_k, where $k = 1, 2, ..., n$ and each three parties A_k, B_k and C_k share one copy of (1). Suppose now that the parties are aligned in a row, such that for each k, the party C_k is located exactly between A_k and B_k, and the parties B_k and A_{k+1} $(k = 1, 2, ..., n-1)$ are close to each other so that they can perform joint measurements on there spatially separated systems. In this scenario, the EoC between the two parties located at the edges of the chain, i.e. A_1 and B_n, is 2 ebits, since the parties can generate 2 ebits between A_k and B_k and then use entanglement swapping to generate maximally entangled state between A_1 and B_n. Now, if we add the restriction that no classical information can be transmitted in the direction from A_k to C_k, than it is impossible to generate a maximally entangled state between A_k and C_k even with some probability. As a result, the maximum average entanglement, when measured by the G-concurrence (see Eqs. (20,21)), that can be generated between A_k and B_k is bounded above by a positive number $c < 1$. From the corollary after theorem 1 in [11], it follows that the maximum G-concurrence that can be generated between A_1 and B_n is bounded from above by c^n. Therefore, in the limit $n \to \infty$ the maximum G-concurrence that can be generated between A_1 and B_n approaches zero; this implies that the entropy of entanglement is bounded by $\log_2 3$ ebits (because zero G-concurrence implies that at least one of the Schmidt coefficients is zero). Hence, we can see a significant advantage of 2-way over 1-way classical channels.

Definitions and notations

The definition of EoC (and LE) corresponds to a family of measures, with each one being parasitic on (i.e. defined in terms of) a different bipartite measure of entanglement. The latter is required to be an entanglement monotone with respect to LOCC operations on the bipartite system. Following [9], we call it the *root entanglement measure* and denote it by E_{Rt}.

Definition 1. Given a *mixed* state of n systems, the localizable entanglement is defined as the maximum average of the root entanglement measure that a distinguished pair of parties (A and B) can share after LOCC by the *other* $n-2$ parties.

Note that since we consider here mixed multipartite systems, even in the asymptotic limit there are many possible choices for the root entanglement measure [27].

Definition 2. Given a *mixed* state of n systems, the entanglement of collaboration is defined as the maximum average of the root entanglement measure that a distinguished pair of parties (A and B) can share after general LOCC by *all* the parties (including A and B).

It is clear from the two definitions above that the EoC$\geq$LE with equality iff the LE is an entanglement monotone. One of the questions we consider in this paper is for which root entanglement measures EoC=LE. We start with some notations.

Let $Y \equiv AB$, be the system onto which entanglement is to be localized, and $Z \equiv C_1C_2\ldots C_{n-2}$ be the system available to the $n-2$ other parties that are trying to assist in the distillation. We denote by ρ^{YZ} the mixed multipartite state shared by the n parties.

Recall that a multipartite measure of entanglement E is an entanglement monotone iff the following two conditions are satisfied [21]:

(1) For any local operation, $\mathcal{E}_k$, performed by one of the parties (in Y or in Z)

$$E(\rho^{YZ}) \geq \sum_k p_k E(\varrho_k^{YZ}) \ , \qquad (2)$$

where $p_k \equiv \mathrm{Tr}\left[\mathcal{E}_k(\rho^{YZ})\right]$ and $\varrho_k^{YZ} \equiv \mathcal{E}_k(\rho^{YZ})/p_k$.
(2) E is a convex function, that is, $E(\rho) \leq \sum_k w_k E(\rho_k)$ for any ensemble $\{w_k, \rho_k\}$ such that $\rho = \sum_k w_k \rho_k$.

A mathematical expression for the localizable entanglement

In order to determine under what conditions the LE, E_{Loc}, satisfies the two conditions above, an explicit expression for $E_{Loc}(\rho^{YZ})$ is needed. This expression is found with the help of the Jamiolkowski isomorphism.

The Jamiolkowski isomorphism states [22]: every density operator ρ^{YZ} is associated with a CP map $\mathcal{J}_\rho : \mathcal{B}(\mathcal{H}^Z) \rightarrow \mathcal{B}(\mathcal{H}^Y)$ such that $\rho^{YZ} = \mathcal{J}_\rho \otimes \mathcal{I}(|\psi^+\rangle\langle\psi^+|)$ where $|\psi^+\rangle = \sum_i |i\rangle \otimes |i\rangle \in \mathcal{H}^Z \otimes \mathcal{H}^Z$ is an unnormalized maximally entangled state. Furthermore, note that for any given operator $\mathcal{A}$, $\mathcal{I} \otimes \mathcal{A} |\psi^+\rangle = \mathcal{A}^t \otimes \mathcal{I} |\psi^+\rangle$, where $\mathcal{A}^t$ is the transpose of $\mathcal{A}$. Given this we deduce the following:

Proposition 1. *The localizable entanglement is given by the expression*

$$E_{Loc}(\rho^{YZ}) = \max_{\{Q_k^Z\}} \sum_k p_k E_{Rt}(\sigma_k^Y) \tag{3}$$

where

$$p_k = \mathrm{Tr}\left[\rho^Z Q_k^Z\right] \tag{4}$$

$$\sigma_k^Y = \frac{\mathcal{J}_\rho(Q_k^Z)}{p_k} \tag{5}$$

where $\rho^Z \equiv \mathrm{Tr}_Y \rho^{YZ}$ and $\mathcal{J}_\rho : \mathcal{B}(\mathcal{H}^Z) \rightarrow \mathcal{B}(\mathcal{H}^Y)$ is the map associated with ρ^{YZ} through the Jamiolkowski isomorphism and where the maximization is over all POVMs $\{Q_k^Z\}$ that can be implemented locally among the parties of Z; thus, $Q_k^Z = Q_k^{C_1} \otimes Q_k^{C_2} \otimes \cdots \otimes Q_k^{C_{n-2}}$.

Note that in Eq. (5) Q_k^Z should be replaced with $\left(Q_k^Z\right)^t$. However, since we take the maximization of all possible POVMs we can drop the transposition sign. With this expression for the LE, we are ready to examine the two conditions (stated above) for monotonicity.

The localizable entanglement is convex

Intuitively, the LE can not increase if one of the subsystems is discarded or if some of the information about the system is lost. Therefore, we would expect that the LE is a convex function as convexity is associated with loss of information [28]. Indeed, as we show below, the LE is convex. Nevertheless, note that for $n = 3$, the LE (in this case called EoA) is a *concave* function when considered as a *bipartite* measure [5].

Proposition 2. *The localizable entanglement is a convex function for* any *root entanglement measure that is an entanglement monotone.*

Proof. We wish to show that if

$$\rho^{YZ} = \sum_l t_l \rho_l^{YZ}, \tag{6}$$

then

$$E_{\mathrm{Loc}}(\rho^{YZ}) \leq \sum_l t_l E_{\mathrm{Loc}}(\rho_l^{YZ}) \tag{7}$$

where the LE, E_{Loc}, is given by (*cf* Eq. (3))

$$E_{\mathrm{Loc}}(\rho^{YZ}) = \max_{\{Q_k^Z\}} \sum_k p_k E_{\mathrm{Rt}}\left[\frac{\mathcal{J}_\rho(Q_k^Z)}{p_k}\right] \tag{8}$$

with E_{Rt} an entanglement monotone on Y, $p_k \equiv \mathrm{Tr}\mathcal{J}_\rho(Q_k^Z)$ and where $\mathcal{J}_\rho : \mathcal{B}(\mathcal{H}^Z) \rightarrow \mathcal{B}(\mathcal{H}^Y)$ is the map associated with ρ^{YZ} through the Jamiolkowski isomorphism. By Eq. (6), $\mathcal{J}_\rho = \sum_l t_l \mathcal{J}_{\rho_l}$ and in particular,

$$\frac{\mathcal{J}_\rho(Q_k^Z)}{p_k} = \sum_l \left(\frac{t_l q_{kl}}{p_k}\right) \frac{\mathcal{J}_{\rho_l}(Q_k^Z)}{q_{kl}}, \tag{9}$$

where $q_{kl} \equiv \mathrm{Tr}\left[\mathcal{J}_{\rho_l}(Q_k^Z)\right]$. But E_{Rt} is an entanglement monotone and therefore also a convex function, so that,

$$E_{\mathrm{Rt}}\left[\frac{\mathcal{J}_\rho(Q_k^Z)}{p_k}\right] \leq \sum_l \left(\frac{t_l q_{kl}}{p_k}\right) E_{\mathrm{Rt}}\left[\frac{\mathcal{J}_{\rho_l}(Q_k^Z)}{q_{kl}}\right]. \tag{10}$$

Combining this result with Eq. (8), we find

$$E_{\mathrm{Loc}}(\rho^{YZ}) \leq \max_{\{Q_{k'}^Z\}} \sum_k \sum_l t_l q_{kl} E_{\mathrm{Rt}}\left[\frac{\mathcal{J}_{\rho_l}(Q_k^Z)}{q_{kl}}\right]. \tag{11}$$

Note however that the right hand side will only be larger if we maximize every element of the sum over l, thus

$$\begin{aligned} E_{\mathrm{Loc}}(\rho^{YZ}) &\leq \sum_l t_l \max_{\{Q_{k'}^Z\}} \sum_k q_{kl} E_{\mathrm{Rt}}\left[\frac{\mathcal{J}_{\rho_l}(Q_k^Z)}{q_{kl}}\right] \\ &= \sum_l t_l E_{\mathrm{Loc}}(\rho_l^{YZ}), \end{aligned} \tag{12}$$

where we have made use of the expression (3) of E_{Loc} for ρ_l^{YZ}. ☐

Sufficient conditions for monotonicity

The result above shows that the convexity requirement for monotonicity is satisfied by the LE for any choice of root entanglement measures. Hence, for a given root entanglement measure, the LE is an entanglement monotone (and therefore equal to the EoC) iff the condition given in Eq. (2) is satisfied. In the theorem below we find sufficient requirements on a root entanglement measure that is generating LE=EoC.

Theorem 3. *The localizable entanglement E_{Loc} is an entanglement monotone whenever the root entanglement measure E_{Rt} satisfies:*
(i) *Homogeneity of degree 1: $E_{Rt}(c\rho) = cE_{Rt}(\rho)$, where c is a positive real number.*
(ii) *There exist a function f such that*
(a) For an arbitrary trace-decreasing completely positive (CP) map, $\mathcal{E}$,

$$E_{Rt}\left[\mathcal{E}\otimes\mathcal{I}^B(\rho^{AB})\right] \le f(\mathcal{E})E_{Rt}(\rho^{AB})$$

for all bipartite states ρ^{AB}.
(b) For any set of trace-decreasing CP maps, $\{\mathcal{E}_j\}$, such that the map $\sum_j \mathcal{E}_j$ is trace-preserving,

$$\sum_j f(\mathcal{E}_j) \le 1.$$

Proof. Given that convexity is already established for the LE, all we need to show is that any local operation performed by one of the parties in Y or Z can not increase on average the LE. Furthermore, note that we need only consider local operations on the two subsystems of Y because it is clear, by the definition of the LE, that it cannot increase under LOCC on Z. Thus, without loss of generality, it is left to show that any local operation performed on subsystem A cannot increase the LE. The local operation performed on A with outcomes $\{j\}$ is described by a set of trace-decreasing completely positive maps $\{\mathcal{E}_j\}$. We need to show that

$$E_{\rm Loc}(\rho^{YZ}) \ge \sum_j q_j E_{\rm Loc}(\varrho_j^{YZ}), \tag{13}$$

where $q_j \equiv {\rm Tr}\left[\mathcal{E}_j(\rho^{YZ})\right]$ and $\varrho_j^{YZ} \equiv \mathcal{E}_j(\rho^{YZ})/q_j$. Note that in this short notation, $\mathcal{E}_j$, stands for $\mathcal{E}_j\otimes\mathcal{I}^B\otimes\mathcal{I}^Z$, where $\mathcal{I}^B$ and $\mathcal{I}^Z$ are the identity maps on B and Z. From the expression given in Eq.(3) for the LE we have

$$\sum_j q_j E_{\rm Loc}(\varrho_j^{YZ}) = \sum_j q_j \max_{\{Q_{k'}^Z\}} \sum_k p_k E_{\rm Rt}\left[\frac{\mathcal{J}_{\varrho_j}(Q_k^Z)}{p_k}\right] . \tag{14}$$

Now, from the Jamiolkowski isomorphism,

$$\begin{aligned}\mathcal{J}_{\varrho_j}\otimes\mathcal{I}^Z\left(|\psi^+\rangle\langle\psi^+|\right) &= \varrho_j^{YZ} = \frac{1}{q_j}\mathcal{E}_j(\rho^{YZ}) \\ &= \left(\frac{1}{q_j}\mathcal{E}_j\circ\mathcal{J}_\rho\right)\otimes\mathcal{I}^Z\left(|\psi^+\rangle\langle\psi^+|\right) ,\end{aligned} \tag{15}$$

so that, $\mathcal{J}_{\varrho_j} = \frac{1}{q_j}\mathcal{E}_j\circ\mathcal{J}_\rho$ (because two local superoperators that have the same image on the maximally entangled state are equivalent). Thus,

$$\begin{aligned}\sum_j q_j E_{\rm Loc}(\varrho_j^{YZ}) &= \sum_j q_j \max_{\{Q_{k'}^Z\}} \sum_k p_k E_{\rm Rt}\left[\frac{\mathcal{E}_j\circ\mathcal{J}_\rho(Q_k^Z)}{q_j p_k}\right] \\ &= \sum_j \max_{\{Q_{k'}^Z\}} \sum_k p_k E_{\rm Rt}\left[\frac{\mathcal{E}_j\circ\mathcal{J}_\rho(Q_k^Z)}{p_k}\right] ,\end{aligned} \tag{16}$$

where the last equality follows from the homogeneity assumption (**i**). Now, by assumption (**ii**), there exist a function f such that

$$\sum_j q_j E_{\rm Loc}(\varrho_j^{YZ}) \le \sum_j \max_{\{Q_{k'}^Z\}} \sum_k p_k f(\mathcal{E}_j) E_{\rm Rt}\left[\frac{\mathcal{J}_\rho(Q_k^Z)}{p_k}\right] . \tag{17}$$

Hence,

$$\begin{aligned}\sum_j q_j E_{\rm Loc}(\varrho_j^{YZ}) &\le \sum_j f(\mathcal{E}_j) \max_{\{Q_{k'}^Z\}} \sum_k p_k E_{\rm Rt}\left[\frac{\mathcal{J}_\rho(Q_k^Z)}{p_k}\right] \\ &= \left[\sum_j f(\mathcal{E}_j)\right] E_{\rm Loc}(\rho^{YZ}) .\end{aligned} \tag{18}$$

By assumption (**ii**)

$$\sum_j f(\mathcal{E}_j) \le 1 , \tag{19}$$

which complete the proof. ☐

The theorem above shows that the LE equals the EoC when the root entanglement measure satisfies conditions (i) and (ii). We now show that there exist at least one measure of entanglement, the concurrence, that satisfies these conditions.

The G-concurrence

Originally, the concurrence has been defined for a pair of qubits [10], though generalization to higher dimensions are possible [24], but not unique [11]. Here we focus on one of the generalizations which we call the G-concurrence [11, 12].

For a pure bipartite state, $|\psi\rangle$, the G-concurrence is defined as the *geometric mean* of the (non-negative) Schmidt numbers

$$G(|\psi\rangle) \equiv d(\lambda_0\lambda_1\cdots\lambda_{d-1})^{\frac{1}{d}} = d\left[{\rm Det}\left(A^\dagger A\right)\right]^{\frac{1}{d}} , \tag{20}$$

where the matrix elements of A are a_{ij} ($|\psi\rangle = \sum_{ij} a_{ij}|i\rangle|j\rangle$). For a mixed $d\times d$-dimensional bipartite state, ρ, the G-concurrence is defined in terms of the convex roof extension:

$$G(\rho) = \min\sum_i p_i G(|\psi_i\rangle) \quad \left(\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|\right) , \tag{21}$$

where the minimum is taken over all decompositions of ρ. In [11] it has been shown that the G-concurrence as defined in Eqs. (20,21) is a bipartite entanglement monotone. It has been shown that it can be interpreted operationally as a kind of entanglement capacity and that it is a computationally manageable measure of entanglement [12]. Very recently, the G-concurrence has been proved fruitful in calculating the average entanglement of

random bipartite pure states [25] and also played a crucial role in a demonstration of an asymmetry of quantum correlations [26].

Corollary 4. *The LE is equal EoC whenever the root entanglement measure is taken to be the G-concurrence.*

The corollary above is surprising taking into account the example in [9] (*cf* Eq. (1)). Note however that the corollary does not contradict the results in [9] since the G-concurrence vanish for $m \times n$ states with $m \neq n$. Furthermore, note that if the parties A and B each hold a qubit then the corollary states that the LE, which has been studied in [7] in terms of the concurrence and in [14, 15, 16, 17, 18, 19] for spin chains, is indeed an entanglement monotone.

Proof. (of corollary 4)
The corollary above follows from two properties of the G-concurrence [11]:

$$G(c|\psi\rangle) = |c|^2 G_d(|\psi\rangle) \tag{22}$$

$$G\left(\hat{A}\otimes\hat{B}|\psi\rangle\right) = \left|\mathrm{Det}\left(\hat{A}\right)\right|^{2/d}\left|\mathrm{Det}\left(\hat{B}\right)\right|^{2/d} G(|\psi\rangle)\ . \tag{23}$$

Hence, condition (i) in theorem 3 is satisfied, as the G-concurrence is homogeneous. We now show that also condition (ii) is satisfied.

Given a bipartite density matrix $\rho^{AB} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ we take $\{p_i,\ |\psi_i\rangle\}$ to be an optimal decomposition such that

$$G(\rho^{AB}) = \sum_i p_i G(|\psi_i\rangle)\ .$$

Let $\{\mathcal{E}_j\}$ be a set of trace-decreasing CP maps, such that the map $\sum_j \mathcal{E}_j$ is trace-preserving. Each map can then be written in the Kraus form:

$$\mathcal{E}_j(\rho^{AB}) = \sum_k M_{jk}^{\mathrm{A}} \otimes \mathcal{I}^{\mathrm{B}} \rho^{AB} M_{jk}^{\mathrm{A}\dagger} \otimes \mathcal{I}^{\mathrm{B}}\ ,$$

where $\sum_{jk} M_{jk}^{\dagger} M_{jk} = \mathcal{I}^{\mathrm{A}}$. Now, we denote the normalized state $|\phi_{jki}\rangle \equiv N_{ikj}^{-1/2} M_{jk}\otimes\mathcal{I}|\psi_i\rangle$, where the normalization factor, N_{ikj}, is taken such that $\langle\phi_{jki}|\phi_{jki}\rangle = 1$. Thus,

$$\mathcal{E}_j(\rho^{AB}) = \sum_{i,k} p_i N_{ikj} |\phi_{jki}\rangle\langle\phi_{jki}|\ ,$$

and since G is defined in terms of the convex roof extension we have

$$G\left(\mathcal{E}_j(\rho^{AB})\right) \le \sum_{i,k} p_i N_{ikj} G(|\phi_{jki}\rangle)$$
$$= \sum_{i,k} p_i G(M_{jk}\otimes\mathcal{I}||\psi_i\rangle) = \sum_k |\mathrm{Det} M_{jk}|^{2/d}\, G(\rho^{\mathrm{AB}})\ ,$$

where we have used the two properties given in Eq. (23) and the optimality of the decomposition $\{p_i,\ \ |\psi_i\rangle\}$. Hence, defining $f(\mathcal{E}_j) \equiv \sum_k |\mathrm{Det} M_{jk}|^{2/d}$ we get $G\left(\mathcal{E}_j(\rho^{AB})\right) \le f(\mathcal{E}_j) G(\rho^{\mathrm{AB}})$ and

$$\sum_j f(\mathcal{E}_j) = \sum_{j,k} |\mathrm{Det} M_{jk}|^{2/d} \le \frac{1}{d}\sum_{j,k} \mathrm{Tr} M_{jk}^{\dagger} M_{jk} = 1\ ,$$

where we have used the geometric-arithmetic inequality. This conclude the proof. □

Summary and conclusions

In summary, following [9], we have introduced the EoC and compared it with the LE. With the help of the Jamiolkowski isomorphism, we were able to find a simple expression for the LE of multipartite mixed states and to prove that it is a convex function. We have also found a set of sufficient conditions for which the EoC equals the LE. We have shown that these conditions are met when the LE and the EoC are measured with the concurrence or with one of its generalizations to higher dimensions called the G-concurrence.

The example given in [9] (*cf* Eq. (1)) shows that the LE fail to be an entanglement monotone (the usual kind, with respect to unrestricted LOCC) when it is defined in terms of a root entanglement measure that can distinguish maximally entangled states from non-maximally entangled states. This left open the possibility that the LE can be an entanglement monotone for choices of the root measure that are not of this sort. Here we have shown that this possibility is indeed realized in the case where the root measure is one of the generalizations of the concurrence. Determining in which Hilbert spaces and for which root entanglement measures the LE is a monotone will help to identify those distributed QIP tasks for which having a collaboration among all parties provides no advantage over merely having the assistance of $n-2$ parties.

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[27] In the asymptotic limit of many copies of a *pure* multipartite state, $E_{\rm Rt}$ is unique and is taken to be the entropy of entanglement as the optimal protocol generates a probability distribution of bipartite *pure* states shared by A and B.
[28] It is interesting to note, however, that it is not always straightforward to equate loss of information with mixing [23].